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A Geometric Approach to Time Evolution Operators of Lie Quantum Systems

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Abstract Lie systems in Quantum Mechanics are studied from a geometric point of view. In particular, we develop methods to obtain time evolution operators of time-dependent Schrödinger equations of Lie type and we show how these methods explain certain *ad hoc* methods used in previous papers in order to obtain exact solutions. Finally, several instances of time-dependent quadratic Hamiltonian are solved.

Keywords Time evolution · Lie systems

1 Introduction

The use of tools of modern differential geometry has been shown to be very useful in many different problems in physics and in particular Lie groups and Lie algebras have played a prominent rôle in the development of Quantum Mechanics. The main concern of Lie was the integration of systems of differential equations admitting infinitesimal symmetries but as a byproduct of his work we have available a lot of relevant tools to deal with many different problems not only in differential geometry and classical mechanics but also in Quantum Mechanics.

Our aim here is to show the efficiency in solving quantum problems of the theory developed by Lie for dealing with systems of differential equations admitting nonlinear superposition rules for solutions and that therefore maybe considered as a generalization of non-autonomous linear systems. More specifically, we will be mainly interested in finding the time evolution operator for quantum systems described by time-dependent quantum Hamiltonians which turn out to be the quantum counterpart of the above mentioned Lie systems of differential equations.

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Time-dependent Schrödinger equations of Lie type are Schrödinger equations of motion for which the solutions of the equations can be obtained from the solutions of an equation in a Lie group G related to the Hamiltonian in a way to be explained in the paper. A particular example is the harmonic oscillator, almost ubiquitous in physics, when the mass and angular frequency are not constant but become real time-dependent parameters [1–6], may be as a consequence of interaction with the environment [7]. Methods for solving such systems have been developed since long time ago, the generalized invariant method proposed by Lewis and Riesenfeld [8] being a typical and powerful method. Generalizations of such time-dependent harmonic oscillators have also been considered and general quadratic systems have been very often studied from different perspectives [9–17].

This kind of equations appears very often in physics: quantum optics [18, 19], quantum chemistry [20], Paul trap [21–24], quantum dissipation [25], fluid dynamics [26], etc. In particular, exact solutions of this type of equations appear many times in the literature [7–28]. The methods used to obtain the solutions of these problems are numerical or reduce the problem of finding out the solution of the time-dependent Schrödinger equation to solve certain differential equations using certain *ad-hoc* hypothesis. This is, for instance, the way used in the Lewis-Riesenfeld method [1, 29] or in the method of unitary transformations [12, 30].

Following the techniques of [31–36] we develop in this paper a geometric interpretation in which the solutions of this sort of Schrödinger equations are obtained in a natural way, by solving a system of differential equations like in other previous papers, but without any additional *ad-hoc* hypothesis. Our presentation is also an important improvement from the algorithmic point of view, it offers information about the difficulty of solving certain Hamiltonians, allows us to solve different physical examples at the same footing and provides us with a frame to explain different methods used in the literature. Finally, our method is an improvement because it shows that Lie's ideas can be applied in the case of Schödinger equations.

The paper is organized as follows: Sect. 2 gives a review of the mathematical theory of Lie systems and summarizes the result of the main theorem due to Lie [31]. Some simple examples of Lie systems are also given and special emphasis is made on the main property, the possibility of relating them with a particular type of equations on a group. Section 3 is devoted to present explicit formulas for a method of solving such equations which is a generalization of the method proposed by Wei and Norman for linear systems [37, 38]. The motion of a classical particle under the action of a linear potential is analysed from this perspective. Lie systems in Quantum Mechanics are studied in Sect. 4 and the reduction method in Sect. 5 where the interaction picture is revisited from a geometric point of view. Some applications of Lie systems in Quantum Mechanics are studied in Sect. 6, and more specifically, time-dependent quadratic Hamiltonians. Other examples of Lie systems appearing in the literature are also pointed out in Sect. 6. Finally, the conclusions and outlook are presented in Sect. 7.

2 Lie Systems of Differential Equations

In this section we will detail some known results about Lie systems that will be applied along this paper. First, we recall that time evolution of many physical systems is described by non-autonomous systems of differential equations

$$\frac{dx^i}{dt} = X^i(x,t), \quad i = 1, \dots, n, \tag{1}$$

for instance, Hamilton equations, or Lagrange equations when transformed to the first order equations by considering momenta or velocities as new variables. In modern geometric terms, such a system is substituted by a t-dependent vector field

$$X=\sum_{i=1}^{n} X^{i}(x,t)\frac{\partial}{\partial x^{i}},$$

whose integral curves satisfy (1).

The theorem of existence and uniqueness of solution for such systems establishes that the initial conditions $x(0)=(x^1(0),\ldots,x^n(0))$ determine the future evolution. It is also well-known that for the simpler case of a homogeneous linear system there is a (linear) function $F:\mathbb{R}^{n^2+n}\to\mathbb{R}^n$, given by

$$x=F(x_{(1)},\ldots,x_{(n)},k_1,\ldots,k_n)=k_1\,x_{(1)}+\cdots+k_n\,x_{(n)}, \tag{2}$$

in such a way that the general solution can be written as a linear combination of n independent particular solutions, $x_{(1)}(t),\ldots,x_{(n)}(t)$,

$$x(t)=F(x_{(1)}(t),\ldots,x_{(n)}(t),k_1,\ldots,k_n)=k_1\,x_{(1)}(t)+\cdots+k_n\,x_{(n)}(t), \tag{3}$$

i.e. $x(t)$ given by (3) is a solution for any choice of $k=(k_1,\ldots,k_n)$ and for each set of initial conditions, the coefficients $k=(k_1,\ldots,k_n)$ can be determined. In a similar way, for an inhomogeneous linear system, there is an affine superposition function $F:\mathbb{R}^{n(n+2)}\to\mathbb{R}^n$ given by

$$\begin{aligned} x&=F(x_{(1)},\ldots,x_{(n+1)},k_1,\ldots,k_n)\\ &=x_{(1)}+k_1(x_{(2)}-x_{(1)})+\cdots+k_n(x_{(n+1)}-x_{(1)}), \end{aligned}\tag{4}$$

and the general solution can be written as the corresponding affine function of $(n+1)$ independent particular solutions

$$\begin{aligned} x(t)&=F(x_{(1)}(t),\ldots,x_{(n+1)}(t),k_1,\ldots,k_n)\\ &=x_{(1)}(t)+k_1(x_{(2)}(t)-x_{(1)}(t))+\cdots+k_n(x_{(n+1)}(t)-x_{(1)}(t)). \end{aligned}\tag{5}$$

Under a non-linear change of coordinates both systems become non-linear ones. However, the fact that the general solution is expressible in terms of a set of particular solutions is maintained, now the superposition function being no longer linear or affine, respectively.

The very existence of such examples of systems of differential equations admitting a non-linear superposition function suggested to Lie the analysis and characterization of such systems. He arrived in this way to the problem of characterizing the systems of differential equations for which a superposition function, allowing to express the general solution in terms of m particular solutions, does exist. The solution of this problem due to Lie [31] asserts that, under very general conditions, such systems are those which can be written as

$$\frac{dx^i}{dt}=b_1(t)\xi^{1i}(x)+\cdots+b_r(t)\xi^{ri}(x),\quad i=1,\ldots,n, \tag{6}$$

where $b_1,\ldots,b_r$, are r functions depending only on t and $\xi^{\alpha i}$, $\alpha=1,\ldots,r$, are functions of $x=(x^1,\ldots,x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by

$$X_\alpha\equiv\sum_{i=1}^{n}\xi^{\alpha i}(x^1,\ldots,x^n)\frac{\partial}{\partial x^i},\quad \alpha=1,\ldots,r, \tag{7}$$

close on a real finite-dimensional Lie algebra, i.e. the vector fields X_α are linearly independent and there exist r^3 real numbers, $f_{\alpha\beta}{}^\gamma$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} f_{\alpha\beta}{}^\gamma X_\gamma. \tag{8}$$

Even if the theorem was not stated with today level of rigour, the result is essentially true. For an intuitive geometric proof see [32], and for a more geometric approach and the uniqueness of such superposition rule see [35, 36].

From the geometric viewpoint, Lie systems are those corresponding to a t-dependent vector field which is a t-dependent linear combination of vector fields

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x),$$

with vector fields X_α closing on a finite-dimensional real Lie algebra. Many of the applications in physics and mathematics of such Lie systems have been developed by Winternitz and coworkers [39–48].

We have mentioned in this section that homogeneous linear systems are Lie systems. Let us consider such a system like

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j, \quad i = 1, \ldots, n, \tag{9}$$

for which

$$X = \sum_{i,j=1}^{n} A^i{}_j(t) x^j \frac{\partial}{\partial x^i}, \tag{10}$$

which is a linear combination with time-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, X_{ij}, \tag{11}$$

of the n^2 vector fields

$$X_{ij} = x^j \frac{\partial}{\partial x^i}, \quad i, j = 1, \ldots, n, \tag{12}$$

for which

$$[X_{ij}, X_{kl}] = \left[x^j \frac{\partial}{\partial x^i}, x^l \frac{\partial}{\partial x^k}\right] = \delta^{il}\, x^j \frac{\partial}{\partial x^k} - \delta^{kj}\, x^l \frac{\partial}{\partial x^i},$$

i.e.

$$[X_{ij}, X_{kl}] = \delta^{il}\, X_{kj} - \delta^{kj}\, X_{il}, \tag{13}$$

which means that the vector fields $\{X_\alpha = X_{ij}, \alpha = (i-1)n + j\}$, with $i, j = 1, \ldots, n$, appearing in the case of a homogeneous system, close on a n^2-dimensional real Lie algebra isomorphic to the $\mathfrak{gl}(n, \mathbb{R})$ algebra. Actually they are the fundamental vector fields corresponding to the natural action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$.

Another remarkable example, with many applications in physics, is that of Riccati equation, which corresponds to $n = 1$ and $m = 3$ (see e.g. [42, 49, 50]).

From the practical viewpoint, the most important property of Lie systems is that as the vector fields X_α appearing in the t-dependent linear combination defining the system

$$X = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha,$$

are assumed to close on a finite-dimensional real Lie algebra $\mathfrak{g}$, if they are complete vector fields, they generate an effective action $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ of a connected Lie group G with Lie algebra $\mathfrak{g}$, see [51], on $\mathbb{R}^n$ and if we determine a curve $g(t)$ in G starting from the neutral element, $g(0) = e$, and such that

$$R_{g^{-1}(t)*g(t)} \dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \equiv a(t), \tag{14}$$

where $\{a_1, \dots, a_r\}$ is a basis of the Lie algebra $\mathfrak{g}$ closing the same commutation relations as the X_α, then, the solution of (6) with initial condition $x(0)$ is given by

$$x(t) = \Phi\big(g(t), x(0)\big).$$

Equation (14) is sometimes written with an abuse of notation as

$$\dot{g} g^{-1} = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \equiv a(t).$$

In this way, the problem of finding the general solution of (6) is reduced to that of determining the above mentioned curve in G solution of (14) and starting from the neutral element [32, 33, 52]. In the particular case of linear systems the group is (a subgroup of) the general linear group $GL(n, \mathbb{R})$ and the action is linear, and therefore, $\Phi(g(t), \cdot)$ provides the time evolution operator.

All the preceding process can be straightforwardly generalized to deal with Lie systems in a general manifold M, the obtained superposition rule being generally only local.

The remarkable point is that once a Lie system with Lie group G in a manifold M which is a homogeneous space for G has been solved on its group G for a curve $a(t)$ in $\mathfrak{g}$ it is possible to obtain the solutions of all other Lie systems with the same Lie algebra $\mathfrak{g}$ and curve $a(t)$ in any other homogeneous space N for G. Moreover, if the Lie group involved in the problem is solvable the problem can be solved by quadratures independently of the curve given in $\mathfrak{g}$. Otherwise, the problem is to be solved, when possible, for each curve in $\mathfrak{g}$ separately, and for some specific examples the solution may be explicitly shown. Of course, if a problem is solved for a certain Lie group G for all curves in $\mathfrak{g}$ the problem is also solved for any problem given by a Lie subgroup H of G.

3 The Wei–Norman Method

In this section we will describe a method to solve directly (14) which is a generalization of the one proposed by Wei and Norman [37, 38] for finding the time evolution operator for a linear systems of type $dU(t)/dt = H(t)U(t)$, with $U(0) = I$ (see also [49]). We will only give here the recipe of how to proceed, the proof can be found for instance in [32, 33, 52].

The generalization of Wei–Norman method consists on writing the solution $g(t)$ of (14) in terms of its second kind canonical coordinates w.r.t. the basis $\{a_1, \ldots, a_r\}$ of the Lie algebra $\mathfrak{g}$, for each value of t, i.e.

$$g(t) = \prod_{\alpha=1}^{r} \exp(-v_\alpha(t)a_\alpha) = \exp(-v_1(t)a_1)\cdots\exp(-v_r(t)a_r), \tag{15}$$

and transforming (14) into a system of differential equations for the unknown functions $v_\alpha(t)$. The curve $g(t)$ we are looking for is the one given by the solution of this last system determined by the initial conditions $v_\alpha(0) = 0$ for all $\alpha = 1, \ldots, r$. The minus signs in the exponentials have been introduced for computational convenience. Now, it can be shown that using the expression (15) and after some mathematical manipulations, (14) becomes the fundamental expression of the Wei–Norman method [52]

$$\sum_{\alpha=1}^{r} \dot{v}_\alpha \left(\prod_{\beta<\alpha} \exp(-v_\beta(t)\mathrm{ad}(a_\beta)) \right) a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha, \tag{16}$$

with $v_\alpha(0) = 0$, $\alpha = 1, \ldots, r$. The resulting system of differential equations for the functions $v_\alpha(t)$ is integrable by quadratures if the Lie algebra is solvable [37, 38], for instance, for nilpotent Lie algebras. Finally, this system of equations depends only on the structure constants of the Lie algebra.

As an interesting example, from the physical point of view, illustrating the possible applications of the theory we can consider the motion of a classical particle under the action of a linear potential. Such example has been studied in [35] and is reproduced here for the sake of completeness. This model, with many applications in physics, has been often considered both in classical and quantum approaches (see, e.g. [53, 54]) and it has recently been studied by Guedes [29]; its solution using the theory of Lie systems was given in [52]. The classical Hamiltonian is

$$H_c = \frac{p^2}{2m} + f(t)x. \tag{17}$$

For instance, when $f(t) = qE_0 + qE\cos\omega t$, it describes the motion of a particle of electric charge q and mass m driven by a monochromatic electric field.

The classical Hamilton equations of motion are

$$\begin{aligned} \dot{x} &= \frac{p}{m}, \\ \dot{p} &= -f(t), \end{aligned} \tag{18}$$

and therefore, the motion is obtained by two quadratures

$$x(t) = x_0 + \frac{p_0 t}{m} - \frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')dt'', \tag{19}$$

$$p(t) = p_0 - \int_0^t f(t')dt'. \tag{20}$$

In the geometric approach, the t-dependent vector field describing the time evolution is

$$X = \frac{p}{m}\frac{\partial}{\partial x} - f(t)\frac{\partial}{\partial p}$$

which can be written as a linear combination $X = \frac{1}{m}X_1 - f(t)X_2$, with

$$X_1 = p\frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial p},$$

being two vector fields closing with $X_3 = \partial/\partial x$ a three-dimensional real Lie algebra isomorphic to the Heisenberg algebra, namely,

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = 0. \tag{21}$$

The flow of these vector fields is given, respectively, by

$$\phi_1(t, (x_0, p_0)) = (x_0 + p_0 t, p_0),$$
$$\phi_2(t, (x_0, p_0)) = (x_0, p_0 + t),$$
$$\phi_3(t, (x_0, p_0)) = (x_0 + t, p_0).$$

In other words, this corresponds to the action of the Lie group of upper triangular 3×3 matrices, the Heisenberg group, on $\mathbb{R}^2$,

$$\begin{pmatrix} \bar{x} \\ \bar{p} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 & \alpha_3 \\ 0 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \\ 1 \end{pmatrix}.$$

Let $\{a_1, a_2, a_3\}$ be a basis of the Lie algebra with non-vanishing defining relations $[a_1, a_2] = -a_3$. Then, the corresponding equation in the group (14) becomes in this case

$$\dot{g}g^{-1} = -\frac{1}{m}a_1 + f(t)a_2.$$

Now, choosing the factorization $g = \exp(-u_3 a_3)\exp(-u_2 a_2)\exp(-u_1 a_1)$ and using the Wei–Norman formula (16) we will arrive to the system of differential equations

$$\dot{u}_1 = \frac{1}{m}, \qquad \dot{u}_2 = -f(t), \qquad \dot{u}_3 - \dot{u}_1 u_2 = 0,$$

together with the initial conditions

$$u_1(0) = u_2(0) = u_3(0) = 0,$$

with solution

$$u_1 = \frac{t}{m}, \qquad u_2 = -\int_0^t f(t')dt', \qquad u_3 = -\frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')dt''.$$

Therefore the motion will be given by

$$\begin{pmatrix} x \\ p \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{m} & -\frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')dt'' \\ 0 & 1 & -\int_0^t f(t')dt' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \\ 1 \end{pmatrix},$$

which reproduces (20). Inverting the matrix we will obtain two constants of the motion corresponding to the values of the initial conditions

$$I_1 = p(t) + \int_0^t f(t')dt',$$
$$I_2 = x(t) - \frac{t}{m}\left(p(t) + \int_0^t f(t')dt'\right)t + \frac{t}{m}\int_0^t dt' \int_0^{t'} f(t'')dt'',$$

the first one being the one given in [29].

4 Schrödinger Lie Systems in Quantum Mechanics

In this section we will review a way to generalise Lie's ideas in order to use them in the case of Schödinger equations [35, 55]. As a new result we show how to apply our method to obtain solutions for quadratic Hamiltonians. Many particular cases of this Hamiltonian can be found in the literature, but obtained by *ad hoc* or approximate methods as it will be explained in detail in a next section. In our method, nevertheless, we found an algorithmic way to find them all.

As far as Quantum Mechanics is concerned, let us first remark that the separable complex Hilbert space of states $\mathcal{H}$ can be seen as a (infinite-dimensional) real manifold admitting a global chart [56]. The Abelian translation group provides us with an identification of the tangent space $T_\phi\mathcal{H}$ at any point $\phi \in \mathcal{H}$ with $\mathcal{H}$ itself, the isomorphism being obtained by associating with $\psi \in \mathcal{H}$ the vector $\dot\psi \in T_\phi\mathcal{H}$ given by

$$\dot\psi f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0},$$

for any $f \in C^\infty(\mathcal{H})$.

Through the identification of $\mathcal{H}$ with $T_\phi\mathcal{H}$ at any $\phi \in \mathcal{H}$ a continuous vector field is just a continuous map $A: \mathcal{H} \to \mathcal{H}$. In particular, a linear operator A on $\mathcal{H}$ is a special kind of vector field. Usually, operators in Quantum mechanics are neither continuous nor defined on the whole space $\mathcal{H}$.

The most relevant case is when A is a skew-self-adjoint operator, $A = -iH$. The reason is that $\mathcal{H}$ can be endowed with a natural (strongly) symplectic structure, and then such skew-self-adjoint operators are singularized as the linear vector fields that are Hamiltonian. The integral curves of such a Hamiltonian vector field $A = -iH$ are the solutions of the corresponding Schrödinger equation [56]. Even when A is not bounded, it can be shown that if A is skew-self-adjoint it must be densely defined and its integral curves are strongly continuous and defined in all $\mathcal{H}$.

On the other side, the skew-self-adjoint operators considered as vector fields are fundamental vector fields relative to the usual action of the unitary group $U(\mathcal{H})$ on the Hilbert space $\mathcal{H}$.

It is important to remark that given a r-dimensional real Lie algebra of skew-self-adjoint operators $-iH_\alpha$

$$[iH_\alpha, iH_\beta] = c_{\alpha\beta}{}^\gamma iH_\gamma, \quad c_{\alpha\beta}{}^\gamma \in \mathbb{R} \tag{22}$$

we can choose a basis $\{a_\alpha \,|\, \alpha = 1, \dots, r\}$ of an abstract Lie algebra $\mathfrak{g}$ isomorphic to that of the X_α such that the Lie brackets of the elements a_α of this Lie algebra, denoted by $[\cdot, \cdot]$

satisfy

$$[a_\alpha, a_\beta] = c_{\alpha\beta}{}^\gamma a_\gamma, \quad c_{\alpha\beta}{}^\gamma \in \mathbb{R}. \tag{23}$$

Now, Lie system theory applies to the case in which a t-dependent Hamiltonian can be written as a linear combination with t-dependent real coefficients of some self-adjoint operators,

$$H(t) = \sum_{\alpha=1}^{r} b_\alpha(t) H_\alpha, \tag{24}$$

where the Hamiltonians H_α are such that the skew-self-adjoint operators $-iH_\alpha$ close a real finite dimensional Lie algebra under the commutator bracket as indicated in (22).

When this happens, we can associate a Lie algebra $\mathfrak{g}$ acting on $\mathcal{H}$ by skew-self-adjoint operators in such a way that there exists a basis $\{a_\alpha\}$ of $\mathfrak{g}$ with $-iH_\alpha$ being the fundamental vector field associated with a_α.

The linear map $X : \mathfrak{g} \to \mathfrak{X}(\mathcal{H})$, $X : a_\alpha \mapsto -iH_\alpha$ is a Lie algebra isomorphism. Then, when $H(t)$ is given by (24), the Schrödinger equation

$$\frac{d\psi}{dt} = -iH(t)\psi = -\sum_{\alpha=1}^{r} b_\alpha(t) i H_\alpha \psi \tag{25}$$

can be reduced to an equation on G like (14) determined by the curve in $\mathfrak{g}$ given by $a(t) \equiv -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha$. Once (14) with initial condition $g(0) = e$ has been solved we can obtain the general solution of the Schrödinger equation.

As a first instance, we can see how the method works in the very simple case of a quantum time-dependent linear potential, which has recently been studied by Guedes [29], instead of using the Lewis and Riesenfeld invariant method [8]. Our method is an improvement with respect to previous ones, because it allows us to obtain the solution in an algorithmic way.

In this case the quantum Hamiltonian is

$$H_q = \frac{P^2}{2m} + f(t)X. \tag{26}$$

In this quantum problem, as pointed out in [35, 57], the quantum Hamiltonian H_q may be written as a sum

$$H_q = \frac{1}{m} H_1 - f(t) H_2,$$

with

$$H_1 = \frac{P^2}{2}, \qquad H_2 = -X.$$

But $-iH_1$ and $-iH_2$ are skew-self-adjoint and close on a four-dimensional Lie algebra with $-iH_3 = -iP$, and $-iH_4 = iI$, which is an extension of the opposite of the Heisenberg Lie algebra (21), i.e. the defining commutation relations are

$$[-iH_1, -iH_2] = -iH_3, \qquad [-iH_1, -iH_3] = 0, \qquad [-iH_2, -iH_3] = -iH_4.$$

Therefore, the Lie algebra of Schrödinger equation given by the Hamiltonian H_q is like in the analogous classical Lie system. Despite of the fact that H_q given by (26) is time-dependent, it is a Lie system and thus we can find the time-evolution operator by solving a related equation on the corresponding Lie group by the Wei–Norman method.

More explicitly, let $\{a_1, a_2, a_3, a_4\}$ be a basis of the Lie algebra with non-vanishing defining relations $[a_1, a_2] = -a_3$ and $[a_2, a_3] = -a_4$. Equation (14) in the group to be considered is now

$$R_{g^{-1}*g}\dot{g} = -\frac{1}{m}a_1 + f(t)\,a_2.$$

In order to find the expression of the wave-function in a simpler way, it is advantageous to use the factorization

$$g = \exp(-v_4\,a_4)\exp(-v_2\,a_2)\exp(-v_3\,a_3)\exp(-v_1\,a_1).$$

In such a case, the Wei–Norman method provides the system

$$\dot{v}_1 = \frac{1}{m}, \qquad \dot{v}_2 = -f(t),$$
$$\dot{v}_3 = \frac{1}{m}v_2, \qquad \dot{v}_4 = -\frac{1}{2m}v_2^2,$$

that, jointly with the initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0$ determines the solution

$$v_1(t) = \frac{t}{m}, \qquad v_2(t) = -\int_0^t dt'\,f(t'), \tag{27}$$

$$v_3(t) = -\frac{1}{m}\int_0^t dt'\int_0^{t'} dt''\,f(t''), \tag{28}$$

$$v_4(t) = -\frac{1}{2m}\int_0^t dt'\left(\int_0^{t'} dt''\,f(t'')\right)^2. \tag{29}$$

Then, applying the evolution operator onto the initial wave-function $\psi(p,0)$, which is assumed to be written in momentum representation, we have

$$\begin{aligned}\psi_t(p) &= U(t,0)\psi_0(p)\\ &= \exp\big(i\,v_4(t)\big)\exp\big(i\,v_2(t)X\big)\exp\big(-i\,v_3(t)P\big)\exp\big(-i\,v_1(t)P^2/2\big)\psi_0(p)\\ &= \exp\big(i\,v_4(t)\big)\exp\big(i\,v_2(t)X\big)e^{-i(v_3(t)p+v_1(t)p^2/2)}\psi_0(p)\\ &= e^{-i(-v_4(t)+v_3(t)(p-v_2(t))+v_1(t)(p-v_2(t))^2/2)}\psi_0\big(p-v_2(t)\big),\end{aligned}$$

where the functions $v_i(t)$ are given by (27), (28) and (29), respectively.

We can proceed in a similar way with the quadratic Hamiltonian in the quantum case, given by [58] (see [35])

$$H_q = \alpha(t)\frac{P^2}{2} + \beta(t)\frac{XP+PX}{4} + \gamma(t)\frac{X^2}{2} + \delta(t)P + \epsilon(t)X + \phi(t)I, \tag{30}$$

where X and P are the position and momentum operators satisfying the commutation relation

$$[X, P] = iI.$$

It is important to solve this quantum quadratic Hamiltonian because of its appearance in many branches of physics.

The Hamiltonian can be written as a sum with t-dependent coefficients

$$H = \alpha(t)H_1 + \beta(t)H_2 + \gamma(t)H_3 - \delta(t)H_4 + \epsilon(t)H_5 + \phi(t)H_6$$

of the Hamiltonians

$$H_1(x,p) = \frac{P^2}{2}, \qquad H_2(x,p) = \frac{1}{4}(XP + PX), \qquad H_3(x,p) = \frac{X^2}{2},$$
$$H_4(x,p) = -P, \qquad H_5(x,p) = X, \qquad H_6(x,p) = I,$$

which satisfy the commutation relations

$$[iH_1, iH_2] = iH_1, \qquad [iH_2, iH_3] = iH_3, \qquad [iH_3, iH_4] = iH_5, \qquad [iH_4, iH_5] = -iH_6,$$
$$[iH_1, iH_3] = 2iH_2, \qquad [iH_2, iH_4] = -\frac{i}{2}H_4, \qquad [iH_3, iH_5] = 0,$$
$$[iH_1, iH_4] = 0, \qquad [iH_2, iH_5] = \frac{i}{2}H_5,$$
$$[iH_1, iH_5] = -iH_4,$$

and $[iH_\alpha, iH_6] = 0$ for $\alpha = 1, \dots, 5$.

This means that the skew-self-adjoint operators $-iH_\alpha$ generate a six-dimensional real Lie algebra. Thus, we can define a basis $\{a_1, \dots, a_6\}$ with the same structure constants as the iH_α with Lie product the commutator of operators.

This six-dimensional real Lie algebra is a central extension of the Lie algebra arising in the classical case by the one-dimensional Lie subalgebra generated by a_6. It is a semidirect sum of the Lie subalgebra $\mathfrak{sl}(2,\mathbb{R})$ spanned by $\{a_1, a_2, a_3\}$ and the Heisenberg–Weyl Lie algebra generated by $\{a_4, a_5, a_6\}$, which is an ideal.

In full similarity with the classical case, in order to find the time-evolution provided by the Hamiltonian (30) we should find the curve $g(t)$ in G such that

$$R_{g^{-1}*g}\dot{g} = -\sum_{\alpha=1}^{6} b_\alpha(t)\, a_\alpha, \qquad g(0) = e,$$

with

$$b_1(t) = \alpha(t), \qquad b_2(t) = \beta(t), \qquad b_3(t) = \gamma(t),$$
$$b_4(t) = -\delta(t), \qquad b_5(t) = \epsilon(t), \qquad b_6(t) = \phi(t).$$

This can be carried out by using the generalized Wei–Norman method, i.e. by writing $g(t)$ in terms of a set of second class canonical coordinates. For instance,

$$g(t) = \exp(-v_4(t)a_4)\exp(-v_5(t)a_5)\exp(-v_6(t)a_6)$$
$$\times \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3) \qquad (31)$$

and a straightforward application of the above mentioned Wei–Norman method technique leads to the system

$$\begin{cases} \dot v_1 = b_1 + b_2\, v_1 + b_3\, v_1^2, & \dot v_4 = b_4 + \frac{1}{2}\, b_2\, v_4 + b_1\, v_5, \\ \dot v_2 = b_2 + 2\, b_3\, v_1, & \dot v_5 = b_5 - b_3\, v_4 - \frac{1}{2}\, b_2\, v_5, \\ \dot v_3 = e^{v_2}\, b_3, & \dot v_6 = b_6 - b_5\, v_4 + \frac{1}{2}\, b_3\, v_4^2 - \frac{1}{2}\, b_1\, v_5^2, \end{cases} \tag{32}$$

with $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = v_6(0) = 0$.

If we consider the following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial v_1} + v_5 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5^2 \frac{\partial}{\partial v_6}, \\ X_2 &= v_1 \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2} v_4 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5 \frac{\partial}{\partial v_5}, \\ X_3 &= v_1^2 \frac{\partial}{\partial v_1} + 2 v_1 \frac{\partial}{\partial v_2} + e^{v_2} \frac{\partial}{\partial v_3} - v_4 \frac{\partial}{\partial v_5} + \frac{1}{2} v_4^2 \frac{\partial}{\partial v_6}, \\ X_4 &= \frac{\partial}{\partial v_4}, \\ X_5 &= \frac{\partial}{\partial v_5} - v_4 \frac{\partial}{\partial v_6}, \\ X_6 &= \frac{\partial}{\partial v_6}, \end{aligned} \tag{33}$$

it is a straightforward computation to see that these vector fields close on the same commutation relations as the corresponding $\{a_\alpha\}$ and thus (32) is a Lie system related with the same Lie group as the Hamiltonian (30) or its corresponding equation in its Lie group.

Now, once the $v_\alpha(t)$ have been determined, the time-evolution of any state will be given by

$$\begin{aligned} |\psi(t)\rangle &= \exp(-v_4(t) i H_4) \exp(-v_5(t) i H_5) \exp(-v_6(t) i H_6) \\ &\quad \times \exp(-v_1(t) i H_1) \exp(-v_2(t) i H_2) \exp(-v_3(t) i H_3) |\psi(0)\rangle \end{aligned}$$

and thus

$$\begin{aligned} |\psi(t)\rangle &= \exp(v_4(t) i P) \exp(-v_5(t) i X) \exp(-v_6(t) i I) \\ &\quad \times \exp\left(-v_1(t) i \frac{P^2}{2}\right) \exp\left(-v_2(t) i \frac{PX + XP}{4}\right) \exp\left(-v_3(t) i \frac{X^2}{2}\right) |\psi(0)\rangle. \end{aligned}$$

5 The Reduction Method in Quantum Mechanics

We start this section with a quick review of the reduction technique explained for example in [33, 50] and then we obtain some new results. First, as a new improvement, while in some previous works like [33, 50] some sufficient conditions where explained to perform a reduction process, here we show that these conditions can be considered as necessary. Also, we will use the reduction technique to explain the interaction picture used in Quantum Mechanics and we will review from the point of view of our theory the method of unitary transformations.

It has been proved in Sect. 2 that the study of Lie systems in homogeneous spaces can be reduced to that of the solution of equations

$$R_{g^{-1}*g}\dot{g} = -\sum_{\alpha=1}^{r} b_\alpha(t)a_\alpha \equiv a(t) \in T_eG \tag{34}$$

with $g(0) = e$.

The reduction method developed in [33] has also shown that given a solution $\tilde{x}(t)$ of a Lie system in a homogeneous space G/H, the solution of the Lie system in the group G, and therefore the general solution in the given homogeneous space, can be reduced to that of a Lie system in the subgroup H. More specifically, if the curve $\tilde{g}(t)$ in G is such that $\tilde{x}(t) = \Phi(\tilde{g}(t), \tilde{x}(0))$ with Φ being the given action of G in the homogeneous space, then $g(t) = \tilde{g}(t)g'(t)$, where $g'(t)$ turns out to be a curve in H which is a solution of a Lie equation in the Lie subgroup H of G. Actually, once the curve $\tilde{g}(t)$ in G has been fixed, the curve $g'(t)$, which takes values in H, satisfies the equation [33]

$$R_{g'^{-1}*g'}\dot{g}' = -\operatorname{Ad}(\tilde{g}^{-1})\left(\sum_{\alpha=1}^{r} b_\alpha(t)a_\alpha + R_{\tilde{g}^{-1}*\tilde{g}}\dot{\tilde{g}}\right) \equiv a'(t) \in T_eH. \tag{35}$$

This transformation law can be understood in the language of connections. It has been shown in [33, 55] that Lie systems can be related with connections in a bundle and that the group of curves in G, which is the group of automorphisms of the principal bundle $G \times \mathbb{R}$ [55], acts on the left on the set of Lie systems in G, and defines an induced action on the set of Lie systems in each homogeneous space for G. More specifically, if $x(t)$ is a solution of a Lie system in a homogeneous space M defined by the curve $a(t)$ in $\mathfrak{g}$, then for each curve $\bar{g}(t)$ in G such that $\bar{g}(0) = e$ we see that $x'(t) = \Phi(\bar{g}(t), x(t))$ is a solution of the Lie system defined by the curve

$$a'(t) = R_{\bar{g}^{-1}*\bar{g}}\dot{\bar{g}} + \operatorname{Ad}(\bar{g})a(t), \tag{36}$$

which is the transformation law for a connection.

In summary, the aim of the reduction method is to find an automorphism $\bar{g}(t)$ such that the right-hand side in (36) belongs to $T_eH \equiv \mathfrak{h}$ for a certain Lie subgroup H of G. In this way, the papers [33, 55] gave a sufficient condition for obtaining this result. In this section we shall study the above geometrical development in Quantum Mechanics and we find out a necessary condition for the right-hand side in (36) to belong to $\mathfrak{h}$.

Lie systems in Quantum Mechanics are those such that

$$H(t) = \sum_{\alpha=1}^{r} b_\alpha(t)H_\alpha, \tag{37}$$

with $-iH_\alpha$ closing under the commutator on a finite dimensional real Lie algebra $\mathfrak{v}$. Therefore, by regarding these operators as fundamental vector fields of an action of a connected Lie group G with Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{v}$, we can relate the Schrödinger equation with a differential equation in G determined by curves in T_eG given by $a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)a_\alpha$ by considering $-iH_\alpha$ as fundamental vector fields of the basis of $\mathfrak{g}$ given by $\{a_\alpha\}$.

Now, the preceding methods allow us to transform the problem into a new one in the same group G, for each choice of the curve $\bar{g}(t)$ but with a new curve $a'(t)$. The action of G on $\mathcal{H}$ is given by an unitary representation U, and therefore the time-dependent vector field determined by the original Hamiltonian H will become a new one with Hamiltonian H'. Its integral curves are the solutions of the equation

$$\frac{d\psi'}{dt} = -iH'(t)\psi' \tag{38}$$

where

$$-iH'(t) = -iU(\bar{g}(t))H(t)U^\dagger(\bar{g}(t)) + \dot{U}(\bar{g}(t))U^\dagger(\bar{g}(t)). \tag{39}$$

That is, from the geometric point of view we have related a Lie system on the Lie group G with certain curve $a(t)$ in T_eG and the corresponding system in $\mathcal{H}$ determined by a unitary representation of G with another one with different curve $a'(t)$ in T_eG and its associated one in $\mathcal{H}$.

Let us choose a basis of T_eG given by $\{c_\alpha \mid \alpha = 1, \dots, r\}$ with $r = \dim \mathfrak{g}$, such that $\{c_\alpha \mid \alpha = 1, \dots, s\}$ be a basis of T_eH, where $s = \dim \mathfrak{h}$, and denote $\{c^\alpha \mid \alpha = 1, \dots, r\}$ the dual basis of $\{c_\alpha \mid \alpha = 1, \dots, r\}$. In order to find $\bar{g}$ such that the right hand term of (36) belongs to T_eH for all t, the condition for $\bar{g}$ is

$$c^\alpha(\mathrm{Ad}(\bar{g})a(t) + R_{\bar{g}^{-1}*\bar{g}}\dot{\bar{g}}) = 0, \quad \alpha = s+1, \dots, r.$$

Now, if θ^α is the left invariant 1-form on G induced from c^α the previous equation implies

$$\theta^\alpha_{\bar{g}^{-1}}\left(R_{\bar{g}^{-1}*e}a(t) - \frac{d\bar{g}^{-1}}{dt}\right) = 0, \quad \alpha = s+1, \dots, r. \tag{40}$$

Let $\tilde{g} = \bar{g}^{-1}$, the last expression implies that $R_{\tilde{g}*e}a(t) - \dot{\tilde{g}}$ is generated by left invariant vector fields on G from the elements of $\mathfrak{h}$. Then, given $\pi^L : G \to G/H$, the kernel of π^L_* is spanned by the left invariant vector fields on G generated by the elements of $\mathfrak{h}$. Then it follows

$$\pi^L_{*\tilde{g}}(R_{\tilde{g}*e}a(t) - \dot{\tilde{g}}) = 0. \tag{41}$$

Therefore, if we use that $\pi^L_* \circ X^R_\alpha = -X^L_\alpha \circ \pi^L$, where X^L_α denotes the fundamental vector field of the action of G in G/H and X^R_α denotes the right-invariant vector field in G whose value in e is a_α, we can prove that $\pi^L(\tilde{g})$ is a solution on G/H of the equation

$$\frac{d\pi^L(\tilde{g})}{dt} = \sum_{\alpha=1}^r b_\alpha(t) X^L_\alpha(\pi^L(\tilde{g})). \tag{42}$$

Thus we obtain that given a certain solution $g'(t)$ in $\mathfrak{h}$ related to the initial $g(t)$ by means of $\tilde{g}(t)$ according to $g(t) = \tilde{g}(t)g'(t)$, then the projection to G/H of $\tilde{g}(t)$ is a solution of (42).

The result so obtained shows that whenever $g'(t)$ is a curve in H then $\tilde{g}(t)$ verifies (42). Moreover, as it has been shown in [33], if $\tilde{g}(t)$ satisfies (42), then $g'(t)$ is a curve in H satisfying (35). The previous result shows that such a condition for obtaining (35) is not only sufficient but necessary too. Thus, we obtain a new result which completes the one found in [33].

Finally, it is worthy of remark that even when this last proof has been developed for Quantum Mechanics, it can be applied also in ordinary differential equations because it appears as a consequence of the group structure of Lie systems which is the same for both the Schrödinger and general differential equations.

5.1 Interaction Picture and Lie Systems

As a first new application of the reduction method for Lie systems, we will analyse in this section how this theory can be applied to explain the interaction picture used in Quantum Mechanics [59]. This picture has been shown to be very effective in the developments of perturbation methods. It plays a rôle when the Hamiltonian can be written as a sum of a simpler Hamiltonian H_1 and another perturbation H_I. In the framework of Lie systems we can analyze what happens when the Hamiltonian is

$$H = H_1 + V(t) = H_1 + \sum_{\alpha=2}^{r} b_\alpha(t) H_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) H_\alpha, \quad b_1(t) = 1, \tag{43}$$

where the set of skew-self-adjoint operators $\{-iH_\alpha \,|\, \alpha = 1, \ldots, r\}$ is closed under commutation and generates a finite dimensional real Lie algebra. The situation is very much similar to the case of control systems with a drift term (here H_1) which are linear in the controls. The functions $b_\alpha(t)$ correspond to the control functions.

According to the theory of Lie systems, let us consider a basis $\{a_\alpha \,|\, \alpha = 1, \ldots, r\}$ of the Lie algebra with corresponding associated fundamental vector fields $-iH_\alpha$. The equation to be studied in T_eG is (34) and if we define $g'(t) = \bar{g}(t)g(t)$, where $\bar{g}(t)$ is a previously chosen curve, it holds a similar equation for $g'(t)$ given by (36).

If in particular we choose $\bar{g}(t) = \exp(a_1 t)$ we find the new equation in T_eG

$$R_{g'^{-1}*g'}\dot{g}' = -\operatorname{Ad}(\exp(a_1 t))\left(\sum_{\alpha=2}^{r} b_\alpha(t) a_\alpha\right) = -\exp(\operatorname{ad}(a_1)t)\left(\sum_{\alpha=2}^{r} b_\alpha(t) a_\alpha\right). \tag{44}$$

Correspondingly, the action of G on $\mathcal{H}$ by a unitary representation defines a transformation of $\mathcal{H}$ in which the state $\psi(t)$ transforms into $\psi'(t) = \exp(iH_1 t)\psi(t)$ and its dynamical evolution is given by the vector field corresponding to the right hand side of (44). In particular, if $\{a_2, \ldots, a_r\}$ span an ideal of the Lie algebra $\mathfrak{g}$ the problem reduces to the corresponding normal subgroup in G.

5.2 The Method of Unitary Transformations

A second application of the theory of Lie systems in Quantum Mechanics and in particular of the reduction method is to obtain information about how to proceed to solve a Lie Hamiltonian in Quantum mechanics. Even when the theoretical results used here can already be found in the literature, the way to proceed can be considered a new improvement.

Every Schrödinger equation of Lie type is determined by a Lie algebra $\mathfrak{g}$, a unitary representation of its connected Lie group G in $\mathcal{H}$ and a curve $a(t)$ in T_eG. Depending on $\mathfrak{g}$ there are two cases. If $\mathfrak{g}$ is solvable we can use the reduction method in Quantum Mechanics to obtain the general solution. If $\mathfrak{g}$ is not solvable, it is not possible to integrate the problem in terms of quadratures in the most general case. But it may be possible to solve a problem completely for some specific curves as for instance it happens for the Caldirola–Kanai Hamiltonian [25]. A way of dealing with such systems is to try to change the curve $a(t)$ into another one $a'(t)$, easier to handle, as it has been done in the previous subsection of the interaction picture. In a more general case than the interaction picture, although any two curves $a(t)$ and $a'(t)$ are always connected by an automorphism, nevertheless, the equation which determines the transformation can be as difficult to be solved as the initial problem. Because of this, it is interesting to look for a curve easy enough to be solved but that we can connect to the initial problem. In any case, we can always express the solution of the initial problem in terms of a solution of the equation determining the transformation. In certain cases, for an appropriate choice of the curve $\bar{g}(t)$ the new curve $a'(t)$ belongs to T_eH for all t, where H is a solvable Lie subgroup of G. In this case we can reduce the problem from $\mathfrak{g}$ to a certain solvable Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Of course, in order to do this, a solution of the equation of reduction is needed, but once this is known we can solve the problem completely in terms of it. Other methods have alternatively been used in the literature, like the Lewis-Riesenfeld (LR) method. However, this method seems to offer a complete solution only if $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is not solvable, the LR method offers a solution which depends on a solution of differential equations, like in the method of reduction.

To sum up, given a Lie system with Lie algebra $\mathfrak{g}$, with G acting unitarily on $\mathcal{H}$ and determined by a curve $a(t)$ in T_eG, the systematic procedure to be used is the following:

- If $\mathfrak{g}$ is solvable we can solve the problem easily by quadratures as it appears in [29, 54],
- if $\mathfrak{g}$ is not solvable, we can try to solve the problem for a given curve like in the Caldirola–Kanai Hamiltonian in [25], by choosing a curve $\bar{g}(t)$ transforming the curve $a(t)$ into another easier to solve, like in the interaction picture. If this does not work we can try to reduce the problem, like in the time-dependent mass and frequency harmonic oscillator or quadratic one-dimensional Hamiltonian in [4, 27, 30, 60] to an integrable case.

6 Applications in Quantum Mechanics

In this section we shall apply our methods to obtain time-dependent evolution operators of several problems found in the physics literature in an algorithmic way. In particular, we will analyze several examples of quadratic Hamiltonians. These examples are studied in the literature under different approaches but here we will study them using the same viewpoint. In our description we will classify the developed examples in terms of whether the related Lie group is either solvable or not. Also, in the non-solvable cases, we will describe some approaches to the study of these Lie systems.

6.1 Solvable Hamiltonians

Quadratic Hamiltonians describe a very large class of physical models. Sometimes, one of these physical models is described by a certain family of quadratic Hamiltonians that can be considered as a Lie system related with a Lie subgroup of the one given for general quadratic Hamiltonians. When this Lie subgroup is solvable the differential equations related with it through the Wei–Norman methods are solvable too and the time-evolution operator can be explicitly obtained. In this subsection we will deal with some instances of this case. In this cases, we can find the explicit solution of these problems already the literature by using for each case a different method.

First, we will fix our attention at the motion of a particle with a time-dependent mass under the action of a time-dependent linear potential term. The Hamiltonian that describe this physical case is

$$H = \frac{P^2}{2m(t)} + S(t)X. \tag{45}$$

The Lie algebra associated with this example is a central extension of the Heisenberg Lie algebra and at the same time, a Lie subalgebra of the one obtained for quadratic Hamiltonians. A basis for the Lie algebra of vector fields related with this physical model is

$$Z_1 = i\frac{P^2}{2}, \qquad Z_2 = iP, \qquad Z_3 = iX, \qquad Z_4 = iI, \tag{46}$$

which closes on a Lie algebra under operator commutation,

$$\begin{aligned} &[Z_1, Z_2] = 0, \qquad [Z_1, Z_3] = 2Z_2, \qquad [Z_1, Z_4] = 0, \\ &[Z_2, Z_3] = Z_4, \qquad [Z_2, Z_4] = 0, \\ &[Z_3, Z_4] = 0. \end{aligned} \tag{47}$$

This Lie algebra is solvable, and then, the related equations obtained through the Wei–Norman method, can be solved by quadratures for any pair of time-dependent coefficients $m(t)$ and $S(t)$. The solution of the associated Wei-Norman system allows us to obtain the time-evolution operator and the wave function solution of the time-dependent Schrödinger equation.

This Hamiltonian has been studied in [61] for some particular cases using *ad-hoc* methods and in general in [54]. Here, we will use the Wei–Norman method. As a quadratic Hamiltonian its equation in the group G is a particular case of the one related with (30)

$$R_{g^{-1}*g}\dot{g} = -\frac{1}{m(t)}a_1 - S(t)a_5 \equiv a_{MS}(t). \tag{48}$$

If we use the factorization given in (31)

$$\begin{aligned} g(t) = {} & \exp(-v_4(t)a_4)\exp(-v_5(t)a_5)\exp(-v_6(t)a_6) \\ & \times \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3) \end{aligned} \tag{49}$$

we can solve the Schrödinger equation by the Wei–Norman method through the set of differential equations (32) for this particular case

$$\begin{aligned} &\dot{v}_1 = \frac{1}{m(t)}, \qquad \dot{v}_2 = 0, \qquad \dot{v}_3 = 0, \\ &\dot{v}_4 = \frac{v_5}{m(t)}, \qquad \dot{v}_5 = S(t), \qquad \dot{v}_6 = -S(t)v_4 - \frac{v_5^2}{2m(t)}, \end{aligned}$$

with initial condition $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = v_6(0) = 0$. The solution of this system can be expressed using quadratures because the related group is solvable:

$$\begin{aligned} v_1(t) &= \int_0^t \frac{du}{m(u)}, \\ v_2(t) &= 0, \\ v_3(t) &= 0, \\ v_4(t) &= \int_0^t \frac{du}{m(u)}\left(\int_0^u S(v)dv\right), \\ v_5(t) &= \int_0^t S(u)du, \\ v_6(t) &= -\int_0^t S(u)\left(\int_0^u \frac{dv}{m(v)}\left(\int_0^v S(w)dw\right)\right)du - \int_0^t \frac{du}{2m(u)}\left(\int_0^u S(v)dv\right)^2 \end{aligned} \tag{50}$$

and the time-evolution operator is obtained by using the last expressions in (49),

$$\begin{aligned} U(g(t)) &= \exp(-v_4(t)iH_4)\exp(-v_5(t)iH_5)\exp(-v_6(t)iH_6)\exp(-v_1(t)iH_1) \\ &= \exp(v_4(t)iP)\exp(-v_5(t)iX)\exp(-v_6(t)iI)\exp(-iv_1(t)P^2/2). \end{aligned} \tag{51}$$

Now, in the case of constant mass, (50) reads as

$$\begin{aligned} v_1(t) &= \frac{t}{m}, \\ v_2(t) &= 0, \\ v_3(t) &= 0, \\ v_4(t) &= \frac{1}{m}\int_0^t \left(\int_0^u S(v)dv\right)du, \\ v_5(t) &= \int_0^t S(u)du, \\ v_6(t) &= -\frac{1}{m}\int_0^t \left(S(u)\int_0^u \left(\int_0^v S(w)dw\right)dv\right)du - \frac{1}{2m}\int_0^t \left(\int_0^u S(v)dv\right)^2 du, \end{aligned}$$

which gives the time-evolution operator if we use them in (51).

Now, as we have obtained the time-evolution operator for a Hamiltonian related with any curve $a_{MS}(t)$ we can consider particular instances of it. For example, for the curves with constant mass m and $S(t) = q\epsilon_0 + q\,\epsilon\,\cos(\omega t)$ studied in [29] we obtain

$$\begin{aligned} &v_1(t) = \frac{t}{m}, \qquad v_2(t) = 0, \qquad v_3(t) = 0, \\ &v_4(t) = \frac{q}{2m\omega^2}\left(2\epsilon + \epsilon_0\omega^2 t^2 - 2\epsilon\cos(\omega t)\right), \qquad v_5(t) = \frac{q}{\omega}(\epsilon_0\omega t + \epsilon\sin(\omega t)), \end{aligned}$$

and

$$\begin{aligned} v_6(t) = \frac{-q^2}{12m\omega^3}&(4\epsilon_0^2\omega^3 t^3 - 3\epsilon(\epsilon - 4\epsilon_0)\omega t \\ &+ 3\epsilon(4\epsilon + 2\epsilon_0(\omega^2 t^2 - 2) - 3\epsilon\cos(\omega t))\sin(\omega t)). \end{aligned} \tag{52}$$

The way to obtain a solution with arbitrary non-constant mass and $S(t) = q\epsilon_0 + q\epsilon\cos(\omega t)$ was pointed out in [29] and solved in [54]. From our point of view, the most general solution is straightforward from (50) because all cases in the literature are particular instances of our approach with general functions $m(t)$ and $S(t)$.

Now, we can obtain the wave function solution of this system. We know that the wave function solution with initial condition ψ_0 is

$$\begin{aligned} \psi_t(x) &= U(g(t))\psi_0(x) \\ &= \exp(iv_6(t))\exp(-v_4(t)iP)\exp(-v_5(t)iX)\exp\left(-v_1(t)i\frac{P^2}{2}\right)\psi_0(x). \end{aligned} \tag{53}$$

However, expressing the initial wave function ψ_0 in the momentum space as $\phi_0(p)$ the solution is given in the same way as before but with $U(g(t))$ in the momentum representation.

In this case the solution with initial condition $\phi_0(p)$ is

$$\begin{aligned}\phi_t(p) &= U(g(t))\phi_0(p)\\ &= \exp(-iv_6(t))\exp(v_4(t)iP)\exp(-v_5(t)iX)\exp\left(-iv_1(t)\frac{P^2}{2}\right)\phi_0(p)\\ &= \exp(-iv_6(t))\exp(v_4(t)iP)\exp(-v_5(t)iX)\exp\left(-iv_1(t)\frac{p^2}{2}\right)\phi_0(p)\\ &= \exp(-iv_6(t))\exp(v_4(t)iP)\exp\left(-iv_1(t)\frac{(p+v_5(t))^2}{2}\right)\phi_0(p+v_5(t))\\ &= \exp\left(-iv_6(t)+iv_4(t)p-iv_1(t)\frac{(p+v_5(t))^2}{2}\right)\phi_0(p+v_5(t)).\end{aligned}$$

6.2 Non-solvable Hamiltonians and Particular Instances

In Sect. 6.1 the differential equations associated to the Quantum Hamiltonians treated there were Lie systems related with a solvable Lie algebra. Thus, by the Wei–Norman method, it has been shown that they were integrable by quadratures. When this does not happen it is not easy to obtain a general solution for the differential equations. Now, we will describe some examples of this type of quadratic Hamiltonians. In general we will not obtain a general solution in terms of the time-dependent functions of the quadratic Hamiltonians. Nevertheless, we will show that for some instances of them the differential equations can be integrated. Note that explicit solutions of these Hamiltonians cannot generally be obtained as our theory explains, but under some integrability conditions on the coefficients [27, 28] the solution can be worked out.

As a first case consider the Hamiltonian of a forced harmonic oscillator with time-dependent mass and frequency given by

$$H=\frac{P^2}{2m(t)}+\frac{1}{2}m(t)\omega^2(t)X^2+f(t)X.$$

This case, either with or without time-dependent frequency, has been studied in [4, 29, 62]. The equations which describe the solutions of this Lie system by the method of Wei–Norman are

$$\begin{aligned}\dot v_1 &= \frac{1}{m(t)}+m(t)\omega^2(t)v_1^2,\\ \dot v_2 &= 2m(t)\omega^2(t)v_1,\\ \dot v_3 &= e^{v_2}m(t)\omega^2(t),\\ \dot v_4 &= \frac{1}{m(t)}v_5,\\ \dot v_5 &= f(t)-m(t)\omega^2(t)v_4,\\ \dot v_6 &= \frac{1}{2}m(t)\omega^2(t)v_4^2-f(t)v_4-\frac{1}{2m(t)}v_5^2,\end{aligned}$$

with initial conditions $v_1(0)=v_2(0)=v_3(0)=v_4(0)=v_5(0)=v_6(0)=0$, where we have used the factorization (31). The solution of this system cannot be obtained by quadratures in the general case because the associated Lie group is not solvable. Nevertheless, we can consider a particular instance of this kind of Hamiltonian, the so called Caldirola–Kanai Hamiltonian [25]. In this case, for the particular time-dependence $m(t)=e^{-rt}m_0$, $\omega(t)=\omega_0$ and $f(t)=0$, the Hamiltonian is given by

$$H=\frac{P^2}{2m_0}e^{rt}+\frac{1}{2}m_0e^{-rt}\omega_0^2X^2. \tag{54}$$

In this case the solution is completely known and is given by

$$\begin{aligned}
v_1(t)&=\frac{2e^{rt}}{m_0(r+\bar{\omega}_0\coth(\frac{t}{2}\bar{\omega}_0))},\\
v_2(t)&=rt+2\log\bar{\omega}_0-2\log\left(r\sinh\left(\frac{t}{2}\bar{\omega}_0\right)+\bar{w}_0\cosh\left(\frac{t}{2}\bar{\omega}_0\right)\right),\\
v_3(t)&=\frac{2m_0\omega_0^2}{r+\bar{\omega}_0\coth(\frac{t}{2}\bar{\omega}_0)},\\
v_4(t)&=0,\qquad v_5(t)=0,\qquad v_6(t)=0,
\end{aligned}$$

where $\bar{\omega}_0=\sqrt{r^2-4\omega_0^2}$. This example shows that the problem may also be solved exactly for particular instances of curves in $\mathfrak{g}$ of Lie systems with non solvable Lie algebras. Another example is the following one

$$H=\frac{P^2}{2m}+\frac{m\omega_0^2}{2(t+k)^2}X^2, \tag{55}$$

for which the solution of the Wei–Norman system reads

$$\begin{aligned}
v_1(t)&=\frac{2(k+t)((k+t)^{\bar{\omega}_0}-k^{\bar{\omega}_0})}{m(k^{\bar{\omega}_0}(\bar{\omega}_0-1)+(k+t)^{\bar{\omega}_0}(\bar{\omega}_0+1))},\\
v_2(t)&=(1+\bar{\omega}_0)\log(k+t)-(1+\bar{\omega}_0)\log k+2\log(2k^{\bar{\omega}_0}\bar{\omega}_0)\\
&\quad-2\log(k^{\bar{\omega}_0}(\bar{\omega}_0-1)+(k+t)^{\bar{\omega}_0}(\bar{\omega}_0+1)),\\
v_3(t)&=\frac{2m\omega_0^2}{k}\frac{(k+t)^{\bar{\omega}_0}-k^{\bar{\omega}_0}}{k^{\bar{\omega}_0}(\bar{\omega}_0-1)+(k+t)^{\bar{\omega}_0}(\bar{\omega}_0+1)},\\
v_4(t)&=0,\qquad v_5(t)=0,\qquad v_6(t)=0,
\end{aligned}$$

where now $\bar{\omega}_0=\sqrt{1-4\omega_0^2}$.

Other examples of Hamiltonians, which can be studied by our method, can be found in [25]. We just mention two examples which can be completely solved

$$\begin{aligned}
H_1&=\frac{P^2}{2m_0}+\frac{1}{2}m_0(U+V\cos(\omega_0t))X^2,\\
H_2&=\frac{P^2}{2m_0}e^{rt}+\frac{1}{2}m_0e^{-rt}\omega_0^2X^2+f(t)X.
\end{aligned}$$

The first one corresponds to a Paul trap [67] as has been studied in [63], and admits a solution in terms of Mathieu's functions. The second one is a damped Caldirola–Kanai Hamiltonian analysed in [61].

6.3 Reduction in Quantum Mechanics

Quite often, when a non-solvable Lie algebra is involved in a quantum problem, it is interesting to solve it in terms of (unknown) solutions of differential equations. Next, we study some examples of how to proceed with the method of reduction in order to deal with problems in this way. So, we will obtain that the reduction method not only can be applied in Quantum Mechanics but also allows us to solve certain problems in an algorithmic way. As far as we know, this kind of application of the theory of reduction in Lie systems to a Schödinger equation is new.

Consider an harmonic oscillator with time-dependent frequency whose Hamiltonian is given by

$$H = \frac{P^2}{2} + \frac{1}{2}\Omega^2(t)X^2. \tag{56}$$

As a particular case of the Hamiltonian described in Sect. 4 this example is related with an equation in the connected Lie group associated to the semidirect sum of $\mathfrak{sl}(2,\mathbb{R})$ with the Heisenberg Lie algebra generated by the ideal $\{a_4, a_5, a_6\}$

$$R_{g^{-1}*g}\dot{g} = -a_1 - \Omega^2(t)a_3, \quad g(0) = e. \tag{57}$$

Since the solution of this equation starts from the identity and $\{a_1, a_2, a_3\}$ close on a $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra, then the Hamiltonian H of (56) is related with the group $SL(2,\mathbb{R})$. Actually this is due to the isomorphism of such a group with the symplectic one, and in higher dimension the group will be $Sp(2n,\mathbb{R})$ instead of $SL(2n,\mathbb{R})$.

As a particular application of the reduction technique we will perform the reduction from $G = SL(2,\mathbb{R})$ to the Lie group related with the Lie subalgebra $\mathfrak{h} = \langle a_1\rangle$. To obtain such a reduction, we have shown in Sect. 5 that we have to solve an equation in G/H, namely

$$\frac{d\pi^L(\tilde{g})}{dt} = \sum_{\alpha=1}^{3} b_\alpha(t) X_\alpha^L(\pi^L(\tilde{g})) \tag{58}$$

where X_α^L are the fundamental vector fields of the action λ of G on G/H. Now, we are going to describe this equation in a set of local coordinates. First, in an open neighborhood U of $e \in G$ we can write in a unique way any element of $SL(2,\mathbb{R})$ as

$$g = \exp(\alpha_3 a_3)\exp(\alpha_2 a_2)\exp(\alpha_1 a_1), \tag{59}$$

where we choose

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad a_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{60}$$

This decomposition allows us to establish a local diffeomorphism between an open neighborhood $V \subset G/H$ and the set of matrices given by $\exp(\alpha_3 a_3)\exp(\alpha_2 a_2)$. Now, the decomposition (59) reads in matrix terms as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}\begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ \phi\theta & \theta^{-1} \end{pmatrix}\begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}. \tag{61}$$

If we express ϕ, θ, ψ in terms of α, β, γ and δ we obtain

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma/\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}. \tag{62}$$

Thus, we can consider the projection $\pi^L : U \subset G \to G/H$ given by

$$\pi^L \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} H \tag{63}$$

which allows to represent locally the elements of G/H as the 2×2 lower triangular matrices with determinant equal to one. Now, given $\lambda_g : g'H \in G/H \to gg'H \in G/H$ as $\lambda_g \circ \pi^L = \pi^L \circ L_g$ then the fundamental vector fields defined in G/H by a_1 and a_3 through the action of G on G/H are given by

$$\begin{aligned} X_1^L(\pi^L(g)) &= \frac{d}{dt}\bigg|_{t=0} \pi^L \left(\exp(-t a_1) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma/\alpha^2 \end{pmatrix}, \\ X_3^L(\pi^L(g)) &= \frac{d}{dt}\bigg|_{t=0} \pi^L \left(\exp(-t a_3) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \end{aligned} \tag{64}$$

and the equation in $V \subset G/H$ is described by

$$\begin{pmatrix} \dot\alpha & 0 \\ \dot\gamma & -\dot\alpha\alpha^{-2} \end{pmatrix} = \begin{pmatrix} -\gamma & 0 \\ \Omega^2(t)\alpha & \gamma\alpha^{-2} \end{pmatrix} \tag{65}$$

and thus we want to obtain a solution of the system

$$\begin{cases} \ddot\alpha = -\Omega^2(t)\alpha, \\ \gamma = -\dot\alpha. \end{cases} \tag{66}$$

Then, if α_1 is a solution of the system (66) the curve $\tilde g(t)$ that verifies $g(t) = \tilde g(t) h(t)$, where $h(t)$ is a solution of an equation defined in the group related with $\mathfrak{h} = \langle a_1 \rangle$, reads

$$\tilde g(t) = \begin{pmatrix} \alpha_1 & 0 \\ -\dot\alpha_1 & \alpha_1^{-1} \end{pmatrix} \tag{67}$$

and the curve which acts on the initial equation in $SL(2, \mathbb{R})$ to transform it into one in the mentioned Lie subalgebra is given by $\bar g(t) = \tilde g^{-1}(t)$,

$$\bar g(t) = \exp(-2 \log \alpha_1\, a_2) \exp\left(-\frac{\dot\alpha_1}{\alpha_1} a_3 \right) = \exp(-\alpha_1 \dot\alpha_1 a_3) \exp(-2 \log \alpha_1\, a_2) \tag{68}$$

and this curve transforms the initial equation in the group given by (57) into the new one given by (cf. (36))

$$a'(t) = -\frac{1}{\alpha_1^2(t)} a_1, \tag{69}$$

which corresponds to the Hamiltonian

$$H' = \frac{1}{2\alpha_1^2(t)} P^2, \tag{70}$$

and the induced transformation in the Hilbert space $\mathcal{H}$ that transforms the initial H into H' is

$$\exp\left(i\frac{\log\alpha_1}{2}(PX+XP)\right)\exp\left(-i\frac{\dot\alpha_1}{2\alpha_1}X^2\right). \tag{71}$$

Both results can be found in [60].

There are other possibilities of choosing different Lie subalgebras of $\mathfrak{g}$ in order to perform the reduction, however the results are always given in terms of a solution of a differential equation.

7 Conclusions and Outlook

It has been shown that the geometric approach to Lie systems of differential equations can be extended to the framework of quantum mechanics and several examples of application have been developed. In particular we have studied time-dependent quadratic Hamiltonians for a single particle and other related models. Some of them are involved in models for dissipative Quantum Mechanics or harmonic oscillators in external fields. In all of these applications we have developed methods to obtain exact solutions that summarize many of the different techniques used in the literature in a simple and unifying framework. The method of reduction has also been revisited and a new theoretical result and some new applications in Quantum Mechanics have been obtained.

But these are not the only applications of the methods developed here, they can be used in many other applications. For example, Hamiltonians of the following type are studied in Quantum Optics [64]

$$H=\omega_1\hat a_1^\dagger\hat a_1+\omega_2\hat a_2^\dagger\hat a_2+\Omega_0(t)(e^{i\omega t}\hat a_1^\dagger\hat a_2+e^{-i\omega t}\hat a_2^\dagger\hat a_1)$$
$$+\Omega_L(t)\hat a_1^\dagger+\Omega_L^*(t)\hat a_1+\Omega_R(t)\hat a_2^\dagger+\Omega_R^*(t)\hat a_2.$$

In this case, it can be shown that this Hamiltonian is a Lie system and thus we may study systems of interacting harmonic oscillators in external fields in our formalism. Many particular cases of this Hamiltonian have been studied in several papers and in different contexts.

Another different example is a system of coupled spins in magnetic fields like the following one found in [66]

$$H=-2A\sum_{i,j}\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j+\mathbf{B}(t)\cdot\sum_i\hat{\mathbf{S}}_\mathbf{i}.$$

In this example our method allows us to obtain exact solutions when $\mathbf{B}(t)$ is a three dimensional vector field in the form

$$\mathbf{B}(t)=(B\sin\theta\cos(\omega t),B\sin\theta\sin(\omega t),B\cos(\theta)).$$

On the other hand, our viewpoint allows us to study important properties of these and other physical systems too [5, 68, 72]. For example, the last example is related with Berry's phases. Our method allows us to obtain the time-evolution operator. Then this information can be used to determine Berry's phases. More specifically, the integrability conditions obtained in [27] can be generalised to solve exactly some Hamiltonians for certain time-dependent coefficients. We can also try to apply control theory to determine the relation of Berry's phases with the field $\mathbf{B}(t)$, study the adiabatic approximation, etc. All these topics are important and will be analyzed in forthcoming papers.

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